

# INVARIANT DIMENSIONS AND MAXIMALITY OF GEOMETRIC MONODROMY ACTION

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**ABSTRACT.** Let  $X$  be a smooth separated geometrically connected variety over  $\mathbb{F}_q$  and  $f : Y \rightarrow X$  a smooth projective morphism. We compare the invariant dimensions of the  $\ell$ -adic representation  $V_\ell$  and the  $\mathbb{F}_\ell$ -representation  $\bar{V}_\ell$  of the geometric étale fundamental group of  $X$  arising from the sheaves  $R^w f_* \mathbb{Q}_\ell$  and  $R^w f_* \mathbb{Z}/\ell\mathbb{Z}$  respectively. These invariant dimension data is used to deduce a maximality result of the geometric monodromy action on  $V_\ell$  whenever  $\bar{V}_\ell$  is semisimple and  $\ell$  is sufficiently large. We also provide examples for  $\bar{V}_\ell$  to be semisimple for  $\ell \gg 0$ .

## 1. INTRODUCTION

Consider a smooth projective  $\mathbb{F}_q$ -morphism  $f : Y \rightarrow X$ , where  $X$  is a smooth separated geometrically connected  $\mathbb{F}_q$ -variety. Fix a geometric point  $\bar{x}_0 : \mathrm{Spec}(\bar{\mathbb{F}}_q) \rightarrow X$ . For any prime  $\ell \nmid q$  and integer  $w$ ,  $\mathcal{F}_\ell := R^w f_* \mathbb{Q}_\ell$  is a *lisso, pure of weight w,  $\mathbb{Q}_\ell$ -sheaf* on  $X$  [De80] inducing an  $\ell$ -adic representation of the *étale fundamental group*  $\pi_1^{et}(X) := \pi_1^{et}(X, \bar{x}_0)$  on the stalk  $\mathcal{F}_{\ell, \bar{x}_0} \cong H^w(Y_{\bar{x}_0}, \mathbb{Q}_\ell) =: V_\ell$ ,

$$(1) \quad \Phi_\ell : \pi_1^{et}(X) \rightarrow \mathrm{GL}(V_\ell);$$

$\bar{\mathcal{F}}_\ell := R^w f_* \mathbb{Z}/\ell\mathbb{Z}$  is a locally constant sheaf on  $X$  inducing an  $\mathbb{F}_\ell$ -representation on the stalk  $\bar{\mathcal{F}}_{\ell, \bar{x}_0} \cong H^w(Y_{\bar{x}_0}, \mathbb{Z}/\ell\mathbb{Z}) =: \bar{V}_\ell$ ,

$$(2) \quad \phi_\ell : \pi_1^{et}(X) \rightarrow \mathrm{GL}(\bar{V}_\ell).$$

The *geometric étale fundamental group* of  $X$ ,  $\pi_1^{et}(X_{\bar{\mathbb{F}}_q}) := \pi_1^{et}(X_{\bar{\mathbb{F}}_q}, \bar{x}_0)$ , is a normal subgroup of  $\pi_1^{et}(X)$  satisfying the exact sequence

$$(3) \quad 1 \rightarrow \pi_1^{et}(X_{\bar{\mathbb{F}}_q}) \rightarrow \pi_1^{et}(X) \rightarrow \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \rightarrow 1$$

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Chun Yin Hui is supported by the National Research Fund, Luxembourg, and cofunded under the Marie Curie Actions of the European Commission (FP7-COFUND).

so that any  $x \in X(\mathbb{F}_q)$  induces a splitting  $i_x$  of (3). The *monodromy group*  $\Gamma_\ell$  (resp.  $\bar{\Gamma}_\ell$ ) and the *geometric monodromy group*  $\Gamma_\ell^{\text{geo}}$  (resp.  $\bar{\Gamma}_\ell^{\text{geo}}$ ) are defined to be the images of  $\pi_1^{et}(X)$  and  $\pi_1^{et}(X_{\bar{\mathbb{F}}_q})$  respectively in  $\text{GL}(V_\ell)$  (resp.  $\text{GL}(\bar{V}_\ell)$ ); their Zariski closures in  $\text{GL}_{V_\ell}$ , denoted respectively by  $\mathbf{G}_\ell$  and  $\mathbf{G}_\ell^{\text{geo}}$ , are called the *algebraic monodromy group* and the *algebraic geometric monodromy group* of  $\Phi_\ell$ .

Since  $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$  is abelian, the geometric monodromy groups  $\Gamma_\ell^{\text{geo}}$  and  $\mathbf{G}_\ell^{\text{geo}}$  are of particular interest by (3). Deligne has proved that the identity component of  $\mathbf{G}_\ell^{\text{geo}}$  is a semisimple subgroup of  $\text{GL}_{V_\ell}$  [De80, Cor. 1.3.9, Thm. 3.4.1(iii)]. Determining  $\Gamma_\ell^{\text{geo}}$  (or  $\bar{\Gamma}_\ell^{\text{geo}}$ ) and  $\mathbf{G}_\ell^{\text{geo}}$  for families of curves (elliptic [Ha08]; hyperelliptic [La90s], [Yu96], [AP07]; trielliptic [AP07]) is of independent interest and also has applications to the arithmetic of function fields (see [Yu96], [Ac06]) and arithmetic geometry (see [Ch97], [Ko06a, Ko06b, Ko06c, Ko08]) over function fields. A crucial point is that for all sufficiently large  $\ell$ , the geometric monodromy  $\Gamma_\ell^{\text{geo}}$  is a *large* compact subgroup of  $\mathbf{G}_\ell^{\text{geo}}(\mathbb{Q}_\ell)$ . The motivation of this paper is to investigate the following large geometric monodromy conjecture. Let  $\pi : \mathbf{G}_\ell^{\text{sc}} \rightarrow \mathbf{G}_\ell^{\text{geo}}$  be the natural morphism such that  $\mathbf{G}_\ell^{\text{sc}}$  is the universal cover of the identity component of  $\mathbf{G}_\ell^{\text{geo}}$ .

**Conjecture 1.** *Let  $\Phi_\ell$  be the  $\ell$ -adic representation defined in (1). Then  $\pi^{-1}(\Gamma_\ell^{\text{geo}})$  is a hyperspecial maximal compact subgroup of  $\mathbf{G}_\ell^{\text{sc}}(\mathbb{Q}_\ell)$  whenever  $\ell$  is sufficiently large.*

Let us make a detour to the characteristic zero case. Suppose  $f : Y \rightarrow X$  is not defined over  $\mathbb{F}_q$ , but over a subfield  $K$  of  $\mathbb{C}$ . Denote the  $w$ th Betti cohomology  $H^w(Y_{\bar{x}_0}(\mathbb{C}), \mathbb{Q})$  by  $V$ , which is acted on by the topological fundamental group  $\pi_1(X(\mathbb{C}))$ . Since the geometric representation  $\Phi_\ell : \pi_1^{et}(X_{\bar{K}}) \rightarrow \text{GL}(V_\ell)$  is arising from  $\Phi : \pi_1(X(\mathbb{C})) \rightarrow \text{GL}(V)$  by the comparison theorem between Betti and étale cohomologies [SGA1, XII], [SGA4, XVI] and the identity component of algebraic monodromy group of  $\Phi$  is semisimple over  $\mathbb{Q}$  [De71, Cor. 4.2.9], the geometric monodromy  $\Gamma_\ell^{\text{geo}}$  is large in  $\mathbf{G}_\ell^{\text{geo}}(\mathbb{Q}_\ell)$  for  $\ell \gg 0$ , thanks to [MVW84]. On the other hand,  $\pi_1^{et}(X)$  satisfies (3) with  $\mathbb{F}_q$  replaced with  $K$ . Since  $\text{Gal}(\bar{K}/K)$  is non-abelian, the monodromy groups  $\Gamma_\ell \subset \mathbf{G}_\ell(\mathbb{Q}_\ell)$  are complicated and carry a lot of arithmetic information. If  $K$  is a number field and  $X = \text{Spec}(K)$ , then  $\Phi_\ell$  is a Galois representation of  $K$  arising from the smooth projective variety  $Y/K$  and the largeness of  $\Gamma_\ell$  in  $\mathbf{G}_\ell(\mathbb{Q}_\ell)$  for  $\ell \gg 0$  follows from the remarkable conjectures of Hodge, Grothendieck, Tate, Mumford-Tate, and Serre [Se94, §11], see also [HL15a, §5]. The prototypical result in this direction is due to Serre [Se72], which states that for any non-CM elliptic curve  $Y$ , the monodromy  $\Gamma_\ell$  on  $H^1$  is  $\text{GL}_2(\mathbb{Z}_\ell)$  for all sufficiently large

$\ell$ , see also [Ri76, Ri85],[Se85],[BGK03, BGK06, BGK10],[Ha11] for certain abelian varieties; [HL15b] for arbitrary abelian varieties; [Se98] for abelian representations; [HL14] for type A representations; and partial results [La95a],[Hu14] for arbitrary varieties. To get large Galois monodromy, one always needs handles on the invariants of  $V_\ell$  and  $\bar{V}_\ell$ . For example when  $Y$  is an abelian variety and  $w = 1$ , Faltings has proved that the Galois invariants of  $V_\ell \otimes V_\ell^*$  and  $\bar{V}_\ell \otimes \bar{V}_\ell^*$  depend essentially on the endomorphism ring  $\text{End}(Y_{\bar{K}})$  if  $\ell$  is sufficiently large [Fa83],[FW84] (the Tate conjecture). Since the Tate conjecture remains largely open, large Galois monodromy is presumably difficult.

Back to our setting  $f : Y \rightarrow X$  over  $\mathbb{F}_q$ , the main idea of this paper is that there is a *cohomological* way, without resorting to the Tate conjecture, to compare the geometric invariant dimensions of  $V_\ell^{\otimes m}$  and  $\bar{V}_\ell^{\otimes m}$  for sufficiently large  $\ell$  and sufficiently many  $m$ .

**Theorem 2.** *For any  $m \in \mathbb{N}$ , if  $\ell$  is sufficiently large, then*

$$\dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}.$$

This is accomplished in §2 first assuming  $X$  is a curve by étale cohomology theory [SGA4, SGA4 $^\frac{1}{2}$ , SGA5],[Mi80],[FK87],[Fu11] and the remarkable theorems of Deligne [De74b, De80], Gabber [Ga83], and de Jong [dJ96], the general case then follows from that by space filling curves [Ka99] and  $\ell$ -independence of  $\mathbf{G}_\ell$  [Ch04].

**Theorem 3.** *If  $\phi_\ell$  is semisimple for all sufficiently large  $\ell$ , then Conjecture 1 holds.*

Theorem 3 is proved in §3 by a recent result of Cadoret and Tamagawa on  $\bar{\Gamma}_\ell^{\text{geo}}$  [CT15], the group theoretic techniques we employed and developed in [HL14], and exploiting the invariant dimension data (Theorem 2 and Corollary 2.3). The  $\mathbb{F}_\ell$ -semisimplicity hypothesis of Theorem 3 holds if  $X$  is a curve and the fibers of  $f$  are curves or abelian varieties [Za74a, Za74b]. It is suggestive that the hypothesis holds in general because the invariant dimensions of  $\Gamma_\ell^{\text{geo}}$  and  $\bar{\Gamma}_\ell^{\text{geo}}$  are alike (Theorem 2) and  $\Gamma_\ell^{\text{geo}}$  is semisimple on  $V_\ell$ . Nevertheless, we provide in §4 some examples for the hypothesis to hold.

## 2. INVARIANT DIMENSIONS

The notation in §1 remains in force. Embed  $\bar{\mathbb{Z}}[\frac{1}{q}]$  into  $\bar{\mathbb{Z}}_\ell$  with unique maximal ideal  $\mathfrak{m}_\ell$ . The common dimension of  $V_\ell$  for all  $\ell$  (not dividing  $q$ ) is also equal to the common dimension of  $\bar{V}_\ell$  for all sufficiently large  $\ell$  [Ga83]. Then whenever  $\dim_{\mathbb{F}_\ell} \bar{V}_\ell = \dim_{\mathbb{Q}_\ell} V_\ell$ , one obtains

$$(4) \quad \dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} \geq \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}$$

for any  $m \in \mathbb{N}$  by identifying  $\Gamma_\ell^{\text{geo}}$  as a subgroup of  $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_\ell))$ , the reduction map  $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_\ell)) \rightarrow \text{GL}(\bar{V}_\ell)$ , and Lemma 2.1.

**Lemma 2.1.** *Let  $F$  be a characteristic 0 non-Archimedean local field with  $\mathcal{O}_F$  the ring of integers. Let  $M$  be a free  $\mathcal{O}_F$ -module of finite rank. If  $W$  is an  $F$ -subspace of  $M \otimes_{\mathcal{O}_F} F$ , then  $W \cap M$  is a direct summand of  $M$ .*

*Proof.* Since  $\mathcal{O}_F$  is a PID and  $\mathcal{O}_F/I$  is finite for any non-zero ideal  $I$ , the finitely generated module  $M/W \cap M$  is a direct sum of a free submodule and a torsion submodule. If  $x \in M$  maps to a torsion element in  $M/W \cap M$ , then  $k \cdot x \in W \cap M$  for some  $k \in \mathbb{N}$ . This implies  $x \in W$  because  $F$  is of characteristic 0. Hence,  $M/W \cap M$  is free and  $W \cap M$  is a direct summand of  $M$ .  $\square$

Let  $d$  be the dimension of  $X$ . Since  $X$  is smooth,  $\mathcal{F}_\ell$  is lisse, and  $\bar{\mathcal{F}}_\ell$  is locally constant, we obtain perfect pairings by Poincaré duality [Mi80, VI Thm. 11.1] which is compatible with the action of *geometric Frobenius*  $\text{Fr}_q$ :

$$(5) \quad \begin{aligned} H^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell) \times H_c^{2d-i}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell^\vee) &\rightarrow \mathbb{Q}_\ell(-d); \\ H^i(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell) \times H_c^{2d-i}(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell^\vee) &\rightarrow \mathbb{F}_\ell(-d). \end{aligned}$$

The geometric invariants admit the following descriptions:

$$(6) \quad \begin{aligned} V_\ell^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} &= H^0(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell); \\ \bar{V}_\ell^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} &= H^0(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell). \end{aligned}$$

Without loss of generality, assume  $x_0$  is an  $\mathbb{F}_q$ -rational point of  $X$  that induces a *splitting* of (3). Then the *multiset*  $A'$  of the  $\text{Fr}_q$ -eigenvalues on  $V_\ell := H^w(Y_{\bar{x}_0}, \mathbb{Q}_\ell)$  are independent of  $\ell$  [De74b] and pure of weight  $w$  [De80]. It follows that the eigenvalues on  $H^0(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell)$  belong to those on  $V_\ell$  by the splitting and (6). One also sees by the same token that the eigenvalues on  $H^0(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell)$  belong to the reduction modulo  $\mathfrak{m}_\ell$  of the eigenvalues on  $V_\ell$  whenever  $\dim_{\mathbb{F}_\ell} \bar{V}_\ell = \dim_{\mathbb{Q}_\ell} V_\ell$ . Define  $A$  to be the following multiset:

$$(7) \quad A := \{q^d \alpha^{-1} : \alpha \in A'\}.$$

We conclude by (5) and above that the numbers in  $A$  are pure of weight  $2d-w$ , the eigenvalues of  $H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell^\vee)$  is a sub-multiset of  $A$ , and the eigenvalues of  $H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell^\vee)$  is a sub-multiset of the reduction modulo  $\mathfrak{m}_\ell$  of  $A$  for  $\ell \gg 0$ .

**Theorem 2.** *For any  $m \in \mathbb{N}$ , if  $\ell$  is sufficiently large, then*

$$\dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}.$$

*Proof.* **Step I.** Assume  $X$  is a (geometrically connected) curve, i.e.,  $d = 1$ . If  $U$  is an affine open subscheme of  $X$  containing  $x_0$ , then  $\pi_1^{et}(U)$  surjects onto  $\pi_1^{et}(X)$  [Fu11, Prop. 3.3.4(i)] and we obtain a commutative diagram:

$$(8) \quad \begin{array}{ccc} \pi_1^{et}(U) & \longrightarrow & \mathrm{GL}(V_\ell) \\ \downarrow & & \downarrow \\ \pi_1^{et}(X) & \longrightarrow & \mathrm{GL}(V_\ell) \end{array}$$

Hence, we may further assume  $X$  is an affine curve.

**Step II.** Let  $e > 0$  be the relative dimension of  $f : Y \rightarrow X$ . Then the dimension of  $Y$  is  $e + 1$ . Let  $Y^c$  be a compactification of  $Y_{\bar{\mathbb{F}}_q}$ . Then  $Y^c$  admits a *simplicial scheme*  $Y$ , projective and smooth over  $\bar{\mathbb{F}}_q$  and an augmentation  $Y \rightarrow Y^c$  which is a *proper hypercovering* of  $Y^c$  (see [dJ96, §1]). This induces a spectral sequence

$$(9) \quad E_1^{i,j} := H^j(Y_i, \mathbb{Z}/\ell\mathbb{Z}) \Rightarrow H^{i+j}(Y^c, \mathbb{Z}/\ell\mathbb{Z})$$

by [Co03, (6.3), Thm. 7.9] (see also [De74a]). Let  $B'$  be the multiset consisting of all the  $\mathrm{Fr}_q$ -eigenvalues on  $H^j(Y_i, \mathbb{Q}_\ell)$  for all  $(i, j) \in \mathbb{Z}_{\geq 0}^2$  satisfying  $i + j = 2(e + 1) - (1 + w)$ . Since  $Y_i$  is smooth projective for all  $i$ , the multiset  $B'$  is mixed of weight  $\leq 2(e + 1) - (1 + w)$  and is independent of  $\ell$  [De74b, De80]. Since there are only finitely many such  $(i, j)$ , the  $\mathrm{Fr}_q$ -eigenvalues on

$$H^{2(e+1)-(1+w)}(Y^c, \mathbb{Z}/\ell\mathbb{Z}) =: H_c^{2(e+1)-(1+w)}(Y_{\bar{\mathbb{F}}_q}, \mathbb{Z}/\ell\mathbb{Z})$$

belong to the reduction (modulo  $\mathfrak{m}_\ell$ ) of  $B'$  for  $\ell \gg 0$  by [Ga83] and the biregular spectral sequence (9). Since  $Y$  is smooth, the reduction (modulo  $\mathfrak{m}_\ell$ ) of the multiset (mixed of weight  $\geq 1 + w$ )

$$B'' := \{q^{e+1}\beta^{-1} : \beta \in B'\}$$

contains all the  $\mathrm{Fr}_q$ -eigenvalues on  $H^{1+w}(Y_{\bar{\mathbb{F}}_q}, \mathbb{Z}/\ell\mathbb{Z})$  for  $\ell \gg 0$  by Poincare duality. Since the spectral sequence

$$E_2^{i,j} := H^i(X_{\bar{\mathbb{F}}_q}, R^j f_* \mathbb{Z}/\ell\mathbb{Z}) \Rightarrow H^{i+j}(Y_{\bar{\mathbb{F}}_q}, \mathbb{Z}/\ell\mathbb{Z})$$

degenerates on page 2 (as  $X$  is an affine curve),  $E_2^{1,w} = H^1(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell)$  is a sub-quotient of  $H^{1+w}(Y_{\bar{\mathbb{F}}_q}, \mathbb{Z}/\ell\mathbb{Z})$ . Thus, the eigenvalues on  $H^1(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell)$  belong to the reduction of  $B''$  for  $\ell \gg 0$ . Then we conclude that the multiset (mixed of weight  $\leq 1 - w$ )

$$(10) \quad B := \{q\beta^{-1} : \beta \in B''\}$$

after reduction contains all the eigenvalues on  $H_c^1(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell^\vee)$  for  $\ell \gg 0$  by  $X$  smooth and Poincare duality again.

**Step III.** By the Lefschetz trace formula on the lisse sheaf  $\mathcal{F}_\ell^\vee$  and the locally constant sheaf  $\bar{\mathcal{F}}_\ell^\vee$  on  $X_{\bar{\mathbb{F}}_q}$  [Mi80, VI Thm. 13.4], we obtain

(11)

$$\begin{aligned} \sum_{x \in X(\mathbb{F}_{q^k})} \text{Tr}(\text{Fr}_q^k : H^w(Y_{\bar{x}}, \mathbb{Q}_\ell)^\vee) &= \sum_{i=0}^2 (-1)^i \text{Tr}(\text{Fr}_q^k : H_c^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell^\vee)); \\ \sum_{x \in X(\mathbb{F}_{q^k})} \text{Tr}(\text{Fr}_q^k : H^w(Y_{\bar{x}}, \mathbb{Z}/\ell\mathbb{Z})^\vee) &= \sum_{i=0}^2 (-1)^i \text{Tr}(\text{Fr}_q^k : H_c^i(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell^\vee)) \end{aligned}$$

for all  $k \in \mathbb{N}$ . Since  $Y_{\bar{x}}$  is smooth projective, the Frobenius action on  $H^w(Y_{\bar{x}}, \mathbb{Z}/\ell\mathbb{Z})^\vee$  factors through  $H^w(Y_{\bar{x}}, \mathbb{Q}_\ell)^\vee$  for  $\ell \gg 0$  [Ga83]. Hence, the reduction of the first local sum is equal to the second local sum for  $\ell \gg 0$ . Since  $H_c^0 = 0$  by  $X$  affine, we obtain

(12)

$$\sum_{i=1}^2 (-1)^i \overline{\text{Tr}(\text{Fr}_q^k : H_c^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell^\vee))} = \sum_{i=1}^2 (-1)^i \text{Tr}(\text{Fr}_q^k : H_c^i(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell^\vee))$$

for  $\ell \gg 0$  by reduction. Denote the reductions of  $A$  (7) and  $B$  (10) by  $\bar{A}$  and  $\bar{B}$  respectively and the following:

- $\{\alpha_1, \dots, \alpha_r\} \subset \bar{A}$  the multiset of the  $\text{Fr}_q$ -eigenvalues on  $H_c^2(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell^\vee)$ ;
- $\{\beta_1, \dots, \beta_s\} \subset \bar{B}$  the multiset of the  $\text{Fr}_q$ -eigenvalues on  $H_c^1(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell^\vee)$ ;
- $\{a_1, \dots, a_t\} \subset \bar{A}$  the multiset of reduction of the  $\text{Fr}_q$ -eigenvalues on  $H_c^2(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell^\vee)$ ;
- $\{b_1, \dots, b_u\}$  the multiset of reduction of the  $\text{Fr}_q$ -eigenvalues on  $H_c^1(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell^\vee)$ .

Note that  $r, t \leq |A|$ ,  $s \leq |B|$ , and the number  $u$  is independent of  $\ell$  [Ka83]. It follows from (12) that the above eigenvalues (in  $\bar{\mathbb{F}}_\ell^*$ ) satisfy

$$(13) \quad a_1^k + \dots + a_t^k + \beta_1^k + \dots + \beta_s^k = \alpha_1^k + \dots + \alpha_r^k + b_1^k + \dots + b_u^k$$

for all  $k \in \mathbb{N}$ . If  $\ell > |A| + \max\{|B|, u\}$ , then by Lemma 2.2 the two multisets coincide:

$$(14) \quad \{a_1, \dots, a_t, \beta_1, \dots, \beta_s\} = \{\alpha_1, \dots, \alpha_r, b_1, \dots, b_u\}.$$

Since  $A$  is pure of weight  $2-w$  and  $B$  is mixed of weight  $\leq 1-w$  (Step II),  $\bar{A} \cap \bar{B} = \emptyset$  for  $\ell \gg 0$  which implies  $t \geq r$  by (14). Together with (4),(5), and (6), we obtain

$$(15) \quad \dim_{\mathbb{F}_\ell} \bar{V}_\ell^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell} V_\ell^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}$$

for all sufficiently large  $\ell$ .

**Step IV.** Since  $f : Y \rightarrow X$  is smooth projective, the natural morphism

$$Y^{[m]} := \overbrace{Y \times_X Y \times_X \cdots \times_X Y}^{\text{$m$ terms}} \rightarrow X$$

is still smooth projective with the fiber

$$(16) \quad (Y^{[m]})_{\bar{x}_0} = \prod^m Y_{\bar{x}_0}$$

inducing the representations  $W_\ell := H^{mw}((Y^{[m]})_{\bar{x}_0}, \mathbb{Q}_\ell)$  and  $\bar{W}_\ell := H^{mw}((Y^{[m]})_{\bar{x}_0}, \mathbb{Z}/\ell\mathbb{Z})$  of  $\pi_1^{et}(X)$ . For all sufficiently large  $\ell$ , we have

$$(17) \quad \dim_{\mathbb{F}_\ell} \bar{W}_\ell^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell} W_\ell^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}$$

by (15). Since the representation  $V_\ell^{\otimes m}$  (resp.  $\bar{V}_\ell^{\otimes m}$ ) is a direct summand of the representation  $W_\ell$  (resp.  $\bar{W}_\ell$ ) by (16) and the Künneth isomorphism, we obtain by (17) that

$$(18) \quad \dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}$$

holds for all sufficiently large  $\ell$ . This proves Theorem 2 when  $X$  is a curve.

**Step V.** For general smooth geometrically connected  $X$ , it suffices to prove Theorem 2 for quasi-projective  $X$  (see (8)). If  $C \subset X$  (containing  $x_0$ ) is a smooth geometrically connected curve over  $\mathbb{F}_q$ , then

$$\Psi_\ell : \pi_1^{et}(C) \rightarrow \mathrm{GL}(V_\ell)$$

factors through  $\Phi_\ell$  for all  $\ell$ . Denote by  $\Lambda_\ell^{\mathrm{geo}}$  and  $\mathbf{H}_\ell^{\mathrm{geo}}$  respectively the geometric monodromy group and the algebraic geometric monodromy group of  $\Psi_\ell$ . Choose  $\ell_0$  such that the dimension of  $\mathbf{G}_{\ell_0}^{\mathrm{geo}}$  is the largest. By [Ka99, Cor. 7, Thm. 8], there exists a space filling curve  $C \subset X$  (smooth, geometrically connected, containing  $x_0$ , over  $\mathbb{F}_q$ ) satisfying

$$(19) \quad \mathbf{H}_{\ell_0}^{\mathrm{geo}} = \mathbf{G}_{\ell_0}^{\mathrm{geo}}.$$

Since the system  $\{\Psi_\ell\}$  is pure of weight  $w$  and is semisimple on the geometric étale fundamental group  $\pi_1^{et}(C_{\bar{\mathbb{F}}_q})$  (Deligne), the identity component  $(\mathbf{H}_\ell^{\mathrm{geo}})^\circ$  (semisimple) is isomorphic to the derived group of the identity component of the algebraic monodromy group of the semisimplification of  $\Psi_\ell$  for all  $\ell$  by (3). This implies  $(\mathbf{H}_\ell^{\mathrm{geo}})^\circ \times \mathbb{C}$  is independent of  $\ell$  by applying [Ch04, Thm. 1.4] to the semisimplification of the system  $\{\Psi_\ell\}$ . In particular, the dimension of  $\mathbf{H}_\ell^{\mathrm{geo}}$  is independent of  $\ell$ . Since we have

$$\dim \mathbf{H}_\ell^{\mathrm{geo}} = \dim \mathbf{H}_{\ell_0}^{\mathrm{geo}} = \dim \mathbf{G}_{\ell_0}^{\mathrm{geo}} \geq \dim \mathbf{G}_\ell^{\mathrm{geo}}$$

and the groups  $\mathbf{H}_\ell^{\mathrm{geo}} \subset \mathbf{G}_\ell^{\mathrm{geo}}$  (by  $\Psi_\ell$  factors through  $\Phi_\ell$ ) have the same number of connected components (by (19) and [LP95, Prop. 2.2(iii)])

for all  $\ell$ , we obtain  $\mathbf{H}_\ell^{\text{geo}} = \mathbf{G}_\ell^{\text{geo}}$  for all  $\ell$ . Since (a)  $\Lambda_\ell^{\text{geo}}$  (resp.  $\Gamma_\ell^{\text{geo}}$ ) is Zariski dense in  $\mathbf{H}_\ell^{\text{geo}}$  (resp.  $\mathbf{G}_\ell^{\text{geo}}$ ) and (b)  $\Psi_\ell$  factors through  $\Phi_\ell$ , we obtain

$$\begin{aligned} \dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} &\stackrel{(b)}{\leq} \dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(C_{\bar{\mathbb{F}}_q})} \stackrel{(18)}{=} \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\pi_1^{et}(C_{\bar{\mathbb{F}}_q})} \\ &\stackrel{(a)}{=} \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\mathbf{H}_\ell^{\text{geo}}(\mathbb{Q}_\ell)} = \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\mathbf{G}_\ell^{\text{geo}}(\mathbb{Q}_\ell)} \stackrel{(a)}{=} \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} \end{aligned}$$

for  $\ell \gg 0$ . We are done by (4).  $\square$

**Lemma 2.2.** *Suppose  $a_1, \dots, a_m, b_1, \dots, b_n \in \bar{\mathbb{F}}_\ell^*$  satisfying  $\max\{m, n\} < \ell$  and*

$$(20) \quad a_1^k + \cdots + a_m^k = b_1^k + \cdots + b_n^k$$

*for all  $1 \leq k \leq \max\{m, n\}$ . Then the two multisets  $\{a_1, \dots, a_m\}$  and  $\{b_1, \dots, b_n\}$  coincide.*

*Proof.* First assume  $m = n$ . Let  $x_1, \dots, x_m$  be indeterminate variables. Denote the elementary symmetric polynomials in  $x_1, \dots, x_m$  by  $e_1, \dots, e_m$  and  $x_1^k + \cdots + x_m^k$  by  $p_k$ . The *Newton's identities* imply

$$e_1, \dots, e_m \in \mathbb{Z}\left[\frac{1}{m!}\right](p_1, \dots, p_m).$$

Hence,  $e_k(a_1, \dots, a_m) = e_k(b_1, \dots, b_m)$  for all  $1 \leq k \leq m$  by (20) and  $m < \ell$ . We conclude that  $\{a_1, \dots, a_m\} = \{b_1, \dots, b_m\}$  by constructing a degree  $m$  polynomial in  $\bar{\mathbb{F}}_\ell[t]$  whose roots are exactly  $a_1, \dots, a_m$  (resp.  $b_1, \dots, b_m$ ).

Suppose  $m > n$ . Let  $b_{n+1}, \dots, b_m$  be all zeros. Then some  $a_i$  is zero by the above case, which contradicts  $a_i \in \bar{\mathbb{F}}_\ell^*$ .  $\square$

**Corollary 2.3.** *For any  $m \in \mathbb{N}$ , if  $\ell$  is sufficiently large, then*

$$\begin{aligned} \dim_{\mathbb{F}_\ell} (\text{Sym}^m \bar{V}_\ell)^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} &= \dim_{\mathbb{Q}_\ell} (\text{Sym}^m V_\ell)^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}; \\ \dim_{\mathbb{F}_\ell} (\text{Alt}^m \bar{V}_\ell)^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} &= \dim_{\mathbb{Q}_\ell} (\text{Alt}^m V_\ell)^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}. \end{aligned}$$

*Proof.* Since the left hand side is always greater than or equal to the right hand side of the equation and the representations  $\bar{V}_\ell^{\otimes m}$  and  $V_\ell^{\otimes m}$  contain respectively  $\text{Sym}^m \bar{V}_\ell$  and  $\text{Sym}^m V_\ell$  (resp.  $\text{Alt}^m \bar{V}_\ell$  and  $\text{Alt}^m V_\ell$ ) as direct summands, the corollary follows from Theorem 2.  $\square$

### 3. MAXIMALITY

If  $X'_{\bar{\mathbb{F}}_q}$  is a connected finite étale cover of  $X_{\bar{\mathbb{F}}_q}$ , then  $\pi_1^{et}(X'_{\bar{\mathbb{F}}_q})$  is a finite index subgroup of  $\pi_1^{et}(X_{\bar{\mathbb{F}}_q})$ . Since  $X'_{\bar{\mathbb{F}}_q} \rightarrow X_{\bar{\mathbb{F}}_q}$  is always defined over some finite extension  $\mathbb{F}_{q'}$  of  $\mathbb{F}_q$  (e.g.,  $X'_{\mathbb{F}_{q'}} \rightarrow X_{\mathbb{F}_{q'}}$ ) which does not affect the geometric monodromy and the restriction of a semisimple

representation to a normal subgroup is still semisimple, it suffices to prove Theorem 3 by considering the base change

$$Y \times_X X'_{\mathbb{F}_{q'}} \rightarrow X'_{\mathbb{F}_{q'}}$$

of  $f : Y \rightarrow X$  by a connected finite Galois étale cover  $X'_{\mathbb{F}_{q'}} \rightarrow X_{\mathbb{F}_{q'}} \rightarrow X$ . Hence, we assume from now on that the algebraic geometric monodromy group  $\mathbf{G}_\ell^{\text{geo}}$  is *connected* for all  $\ell$  by taking a connected finite étale cover of  $X$  [LP95, Prop. 2.2(ii)]. Let  $n$  be the common dimension of  $V_\ell$  for all  $\ell$ , which is also the common dimension of  $\bar{V}_\ell$  for  $\ell \gg 0$ .

**Theorem 3.** *If  $\phi_\ell$  is semisimple for all sufficiently large  $\ell$ , then Conjecture 1 holds.*

*Proof.* **Step I.** For any subgroup  $\bar{\Gamma}$  of  $\text{GL}_n(\mathbb{F}_\ell)$ , denote by  $\bar{\Gamma}^+$  the (normal) subgroup of  $\bar{\Gamma}$  that is generated by  $\bar{\Gamma}[\ell]$ , the subset of order  $\ell$  elements of  $\bar{\Gamma}$ . By taking some connected finite Galois étale cover of  $X$ , we may assume  $\bar{\Gamma}_\ell^{\text{geo}} = (\bar{\Gamma}_\ell^{\text{geo}})^+$  [CT15, Prop. 3.2, Thm. 1.1],  $\bar{\Gamma}_\ell^{\text{geo}}$  is semisimple on  $\bar{V}_\ell$ , and  $\mathbf{G}_\ell^{\text{geo}}$  is connected for all sufficiently large  $\ell$ . Since  $n = \dim_{\mathbb{F}_\ell} \bar{V}_\ell$  for  $\ell \gg 0$ , there exists an *exponentially generated* subgroup  $\bar{\mathbf{S}}_\ell$  of  $\text{GL}_{\bar{V}_\ell}$  such that  $\bar{\Gamma}_\ell^{\text{geo}} = \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)^+$  for all  $\ell \gg 0$  by Nori [No87, Thm. B]. The Nori subgroup  $\bar{\mathbf{S}}_\ell$  is connected and an extension of semisimple by unipotent. Since  $\bar{\Gamma}_\ell^{\text{geo}}$  is semisimple on  $\bar{V}_\ell$  for  $\ell \gg 0$ ,  $\bar{\mathbf{S}}_\ell$  is connected semisimple for  $\ell \gg 0$ . Let  $\bar{\mathbf{S}}_\ell^{\text{sc}} \rightarrow \bar{\mathbf{S}}_\ell$  be the universal covering of  $\bar{\mathbf{S}}_\ell$ . The representation

$$\bar{\mathbf{S}}_\ell^{\text{sc}} \times \bar{\mathbb{F}}_\ell \rightarrow \bar{\mathbf{S}}_\ell \times \bar{\mathbb{F}}_\ell \hookrightarrow \text{GL}_{\bar{V}_\ell \times \bar{\mathbb{F}}_\ell}$$

can be lifted to a representation of some simply-connected *Chevalley scheme* over  $\mathbb{Z}$  for all  $\ell \gg 0$  [Se86] (see [EHK12, Thm. 27]),

$$(21) \quad \rho_{\ell, \mathbb{Z}} : \mathbf{H}_{\ell, \mathbb{Z}} \rightarrow \text{GL}_{V_\mathbb{Z}}.$$

**Step II.** We would like to study the invariants of  $\bar{\mathbf{S}}_\ell$  on  $\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell$ . Let us recall the construction of  $\bar{\mathbf{S}}_\ell$ . Define  $\exp(x)$  and  $\log(x)$  by

$$(22) \quad \exp(x) = \sum_{i=0}^{\ell-1} \frac{x^i}{i!} \quad \text{and} \quad \log(x) = - \sum_{i=1}^{\ell-1} \frac{(1-x)^i}{i}.$$

For all sufficiently large  $\ell$ ,  $\bar{\mathbf{S}}_\ell$  is the Zariski closure in  $\text{GL}_{\bar{V}_\ell} \cong \text{GL}_{n, \mathbb{F}_\ell}$  of the subgroup generated by the one-parameter subgroup

$$(23) \quad t \mapsto x^t := \exp(t \cdot \log(x))$$

for all  $x \in \bar{\Gamma}_\ell^{\text{geo}}[\ell]$  (the order  $\ell$  elements) [No87]. When  $\ell > n$ ,  $x$  is unipotent and  $\log(x)$  is nilpotent by (22). Identify  $\bar{V}_\ell \otimes \bar{\mathbb{F}}_\ell$  with  $\bar{\mathbb{F}}_\ell^n$ , then every entry of the matrix  $x^t \in \text{GL}_n(\bar{\mathbb{F}}_\ell[t])$  is a polynomial of degree less than  $n^2$  by (22) and (23). Similarly, the action of  $x^t$

on  $\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell$  can be identified with an element of  $\mathrm{GL}_{n^m}(\bar{\mathbb{F}}_\ell[t])$  whose entries are polynomials of degree less than  $n^2m$ . Consider an invariant  $v \in (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = (\bar{V}_\ell^{\otimes m})^{\bar{\Gamma}_\ell^{\text{geo}}}$ , then the equation in  $\bar{\mathbb{F}}_\ell[t]^{n^m}$  below

$$x^t \cdot v = v$$

has at least  $\ell$  distinct roots  $t = 0, 1, \dots, \ell - 1$  because  $\mathrm{id}, x, \dots, x^{\ell-1} \in \bar{\Gamma}_\ell^{\text{geo}}$ . This implies  $x^t \cdot v \equiv v$  when  $\ell \geq n^2m$ . Hence, we obtain  $v \in (\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\mathbf{S}}_\ell}$  when  $\ell \geq n^2m$  by the construction of  $\bar{\mathbf{S}}_\ell$ . Since  $\bar{\Gamma}_\ell^{\text{geo}} = (\bar{\Gamma}_\ell^{\text{geo}})^+$  [CT15] is a subgroup of  $\bar{\mathbf{S}}_\ell$  for  $\ell \gg 0$ , we obtain

$$\dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\bar{\mathbb{F}}_\ell} (\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\mathbf{S}}_\ell}$$

for  $\ell \gg 0$ . It follows that

$$\dim_{\mathbb{F}_\ell} ((\oplus^n \bar{V}_\ell)^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\bar{\mathbb{F}}_\ell} ((\oplus^n \bar{V}_\ell)^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\mathbf{S}}_\ell}$$

for  $\ell \gg 0$ . By Corollary 2.3 and embedding  $\mathbb{Q}_\ell$  into  $\mathbb{C}$ , we conclude for  $\ell \gg 0$  that

$$(24) \quad \dim_{\mathbb{C}} (\mathrm{Sym}^m(\oplus^n V_\ell) \otimes \mathbb{C})^{\mathbf{G}_\ell^{\text{geo}}} = \dim_{\bar{\mathbb{F}}_\ell} (\mathrm{Sym}^m(\oplus^n \bar{V}_\ell) \otimes \bar{\mathbb{F}}_\ell)^{\bar{\mathbf{S}}_\ell}.$$

**Step III.** Denote the base change of (21) to  $\mathbb{C}$  by  $\rho_{\ell, \mathbb{C}} : \mathbf{H}_{\ell, \mathbb{C}} \rightarrow \mathrm{GL}_{V_{\mathbb{C}}}$ . For fixed  $m \in \mathbb{N}$ , we obtain by Step I and (24) that for  $\ell \gg 0$ ,

$$(25) \quad \dim_{\mathbb{C}} (\mathrm{Sym}^m(\oplus^n V_\ell) \otimes \mathbb{C})^{\mathbf{G}_\ell^{\text{geo}}} = \dim_{\mathbb{C}} (\mathrm{Sym}^m(\oplus^n V_{\mathbb{C}}))^{\mathbf{H}_{\ell, \mathbb{C}}}.$$

Since there are finitely many connected semisimple subgroup of  $\mathrm{GL}_{n, \mathbb{C}}$  (up to isomorphism), (25) holds for all  $m \in \mathbb{N}$  when  $\ell$  is sufficiently large. Identify  $V_\ell \otimes \mathbb{C}$  with  $V_{\mathbb{C}}$ . Then the (Noetherian) graded rings

$$R = \mathbb{C}[\oplus^n V_{\mathbb{C}}]^{\mathbf{G}_\ell^{\text{geo}}} \quad \text{and} \quad R' = \mathbb{C}[\oplus^n V_{\mathbb{C}}]^{\mathbf{H}_{\ell, \mathbb{C}}}$$

have the same *Hilbert polynomial*, hence the same *Krull dimension* for  $\ell \gg 0$ . Since  $\dim_{\mathrm{Krull}} R = n^2 - \dim \mathbf{G}_\ell^{\text{geo}}$  and  $\dim_{\mathrm{Krull}} R' = n^2 - \dim \rho_{\ell, \mathbb{C}}(\mathbf{H}_{\ell, \mathbb{C}})$  (for example [LP90, §0]), we conclude by the lifting (21) that for all  $\ell \gg 0$ ,

$$(26) \quad \dim \mathbf{G}_\ell^{\text{geo}} = \dim \bar{\mathbf{S}}_\ell.$$

**Step IV.** Suppose  $\ell \geq 5$ . For any compact subgroup  $\Gamma \subset \mathrm{GL}_n(\mathbb{Q}_\ell)$  (resp.  $\bar{\Gamma} \subset \mathrm{GL}_n(\bar{\mathbb{F}}_\ell)$ ), we defined the  $\ell$ -dimension  $\dim_\ell \Gamma$  (resp.  $\dim_\ell \bar{\Gamma}$ ) in [HL14, §2] satisfying the following properties:

- (i)  $\dim_\ell$  is additive on short exact sequences;
- (ii)  $\dim_\ell$  vanishes for pro-solvable groups and finite simple groups that are not of Lie type in characteristic  $\ell$ ;
- (iii) if  $\bar{\Gamma}$  is a finite simple group of Lie type in characteristic  $\ell$ , then there exists some connected adjoint semisimple group  $\bar{\mathbf{S}}/\bar{\mathbb{F}}_\ell$  such that  $\bar{\Gamma}$  is isomorphic to the derived group of  $\bar{\mathbf{S}}(\bar{\mathbb{F}}_\ell)$  and we define  $\dim_\ell \bar{\Gamma} := \dim \bar{\mathbf{S}}$ .

We obtain for  $\ell \gg 0$  that

$$(27) \quad \dim_{\ell} \bar{\Gamma}_{\ell}^{\text{geo}} = \dim_{\ell} \bar{\mathbf{S}}_{\ell}(\mathbb{F}_{\ell})^+ = \dim_{\ell} \bar{\mathbf{S}}_{\ell}(\mathbb{F}_{\ell}) = \dim \bar{\mathbf{S}}_{\ell} = \dim \mathbf{G}_{\ell}^{\text{geo}}$$

by Step I, [No87, 3.6(v)], [HL14, Prop. 3(iii)], and (26) respectively for each equality. Recall the universal covering  $\pi : \mathbf{G}_{\ell}^{\text{sc}} \rightarrow \mathbf{G}_{\ell}^{\text{geo}}$ . Since (a) the kernel and cokernel of  $\pi^{-1}(\Gamma_{\ell}^{\text{geo}}) \rightarrow \Gamma_{\ell}^{\text{geo}}$  are abelian and (b) the kernel of  $\Gamma_{\ell}^{\text{geo}} \twoheadrightarrow \bar{\Gamma}_{\ell}^{\text{geo}}$  is pro-solvable (via the reduction map  $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_{\ell})) \rightarrow \text{GL}(\bar{V}_{\ell})$  for  $\ell \gg 0$ ), we obtain by the properties of  $\dim_{\ell}$  that for  $\ell \gg 0$ ,

$$(28) \quad \dim_{\ell} \pi^{-1}(\Gamma_{\ell}^{\text{geo}}) \stackrel{(a)}{=} \dim_{\ell} \Gamma_{\ell}^{\text{geo}} \stackrel{(b)}{=} \dim_{\ell} \bar{\Gamma}_{\ell}^{\text{geo}} \stackrel{(27)}{=} \dim \mathbf{G}_{\ell}^{\text{geo}} =: g.$$

**Step V.** Let  $\Delta_{\ell}$  be a maximal compact subgroup of  $\mathbf{G}_{\ell}^{\text{sc}}(\mathbb{Q}_{\ell})$  that contains  $\pi^{-1}(\Gamma_{\ell}^{\text{geo}})$ . By [Ti79, 3.2],  $\Delta_{\ell}$  is the stabilizer  $\mathbf{G}_{\ell}^{\text{sc}}(\mathbb{Q}_{\ell})^x$  of a vertex  $x$  in the *Bruhat-Tits building* of  $\mathbf{G}_{\ell}^{\text{sc}}/\mathbb{Z}_{\ell}$ . There exists a smooth affine group scheme  $\mathcal{G}$  over  $\mathbb{Z}_{\ell}$  and an isomorphism  $\iota$  from the generic fiber of  $\mathcal{G}$  to  $\mathbf{G}_{\ell}^{\text{sc}}$  such that  $\iota(\mathcal{G}(\mathbb{Z}_{\ell})) = \mathbf{G}_{\ell}^{\text{sc}}(\mathbb{Q}_{\ell})^x$  [Ti79, 3.4.1]. As  $\mathbf{G}_{\ell}^{\text{sc}}$  is simply-connected semisimple, the special fiber  $\mathcal{G}_{\mathbb{F}_{\ell}}$  is connected [Ti79, 3.5.2]. The maximal compact subgroup  $\Delta_{\ell}$  is *hyperspecial* if and only if  $\mathcal{G}_{\mathbb{F}_{\ell}}$  is reductive [Ti79, 3.8.1], in which case it has the same root datum as the generic fiber [SGA3, XXII, 2.8]. Since (c) the kernel of the reduction map  $r : \mathcal{G}(\mathbb{Z}_{\ell}) \rightarrow \mathcal{G}(\mathbb{F}_{\ell})$  is pro-solvable and (d)  $\mathcal{G}$  is smooth over  $\mathbb{Z}_{\ell}$ , we obtain by the properties of  $\dim_{\ell}$  that for  $\ell \gg 0$ ,

$$(29) \quad \dim_{\ell} r(\pi^{-1}(\Gamma_{\ell}^{\text{geo}})) \stackrel{(c)}{=} \dim_{\ell} \pi^{-1}(\Gamma_{\ell}^{\text{geo}}) \stackrel{(28)}{=} g = \dim \mathbf{G}_{\ell}^{\text{sc}} \stackrel{(d)}{=} \dim \mathcal{G}_{\mathbb{F}_{\ell}}.$$

Since the special fiber  $\mathcal{G}_{\mathbb{F}_{\ell}}$  is connected,  $\mathcal{G}_{\mathbb{F}_{\ell}}$  is semisimple for  $\ell \gg 0$  by (29) and [HL14, Thm. 4(iv)]. It follows from above that  $\Delta_{\ell}$  is hyperspecial and  $\mathcal{G}_{\mathbb{F}_{\ell}}$  is simply-connected semisimple for  $\ell \gg 0$ . For any connected algebraic group  $\bar{\mathbf{G}}$  of dimension  $g$  defined over  $\mathbb{F}_{\ell}$ , the order of  $\bar{\mathbf{G}}(\mathbb{F}_{\ell})$  satisfies

$$(30) \quad (\ell - 1)^g \leq |\bar{\mathbf{G}}(\mathbb{F}_{\ell})| \leq (\ell + 1)^g$$

by [No87, Lem 3.5]. Hence, there exists a constant  $c(g) \geq 1$  depending only on  $g$  such that for  $\ell \gg 0$ ,

$$(31) \quad \frac{(\ell - 1)^g}{c(g)} \leq |r(\pi^{-1}(\Gamma_{\ell}^{\text{geo}}))| \stackrel{\text{subgp}}{\leq} |\mathcal{G}(\mathbb{F}_{\ell})| \stackrel{(30)}{\leq} (\ell + 1)^g,$$

where the first inequality follows by considering (29), (30), the properties of  $\dim_{\ell}$  in Step IV, and the orders of finite simple groups of Lie type in characteristic  $\ell$  [St67, §9] (e.g.,  $|\text{PSL}_k(\ell)| = \frac{1}{(k,\ell-1)} \ell^{k(k-1)/2} (\ell^2 - 1)(\ell^3 - 1) \cdots (\ell^k - 1)$ ). Since  $g := \dim \mathbf{G}_{\ell}^{\text{geo}} \leq n^2$  for all  $\ell$ , the index  $[\mathcal{G}(\mathbb{F}_{\ell}) : r(\pi^{-1}(\Gamma_{\ell}^{\text{geo}}))] \leq C(n)$  (a constant depending only on  $n$ ) for  $\ell \gg 0$ . Since  $\mathcal{G}_{\mathbb{F}_{\ell}}$  is simply-connected semisimple,  $\mathcal{G}(\mathbb{F}_{\ell})$  is generated

by the subset of order  $\ell$  elements  $\mathcal{G}(\mathbb{F}_\ell)[\ell]$  when  $\ell \gg 0$  (see the proof of [HL14, Thm. 4]). Since  $\mathcal{G}(\mathbb{F}_\ell)[\ell]$  belongs to  $r(\pi^{-1}(\Gamma_\ell^{\text{geo}}))$  for  $\ell \gg C(n)$ , the equality  $r(\pi^{-1}(\Gamma_\ell^{\text{geo}})) = \mathcal{G}(\mathbb{F}_\ell)$  holds for  $\ell \gg C(n)$ . Therefore, the subgroup  $\pi^{-1}(\Gamma_\ell^{\text{geo}}) \subset \mathcal{G}(\mathbb{Z}_\ell)$  surjects onto  $\mathcal{G}(\mathbb{F}_\ell)$  under the reduction map  $r$  for  $\ell \gg 0$ . By the main theorem of [Va03], this implies  $\pi^{-1}(\Gamma_\ell^{\text{geo}}) = \mathcal{G}(\mathbb{Z}_\ell) = \Delta_\ell$  for  $\ell \gg 0$ , which is hyperspecial maximal compact in  $\mathbf{G}_\ell^{\text{geo}}(\mathbb{Q}_\ell)$ .  $\square$

**Corollary 3.1.** *For all sufficiently large  $\ell$ , the identity component of the algebraic geometric monodromy group  $\mathbf{G}_\ell^{\text{geo}}$  is unramified over  $\mathbb{Q}_\ell$ .*

*Proof.* Since a connected reductive group  $\mathbf{G}/\mathbb{Q}_\ell$  is unramified if and only if  $\mathbf{G}(\mathbb{Q}_\ell)$  contains a hyperspecial maximal compact subgroup [Mi92, §1],  $\mathbf{G}_\ell^{\text{sc}}$  is unramified for  $\ell \gg 0$  by Theorem 3. Since  $\pi : \mathbf{G}_\ell^{\text{sc}} \twoheadrightarrow (\mathbf{G}_\ell^{\text{geo}})^\circ$  is surjective, the identity component  $(\mathbf{G}_\ell^{\text{geo}})^\circ$  is unramified for  $\ell \gg 0$ .  $\square$

**Remark 3.2.** *Assuming  $\Phi_\ell$  is semisimple for all  $\ell$ , then  $\mathbf{G}_\ell^\circ \times \mathbb{C} \subset \text{GL}_{V_\ell \times \mathbb{C}}$  is independent of  $\ell$  by [Ch04] and [Ka99] (see Step V of Theorem 2). Corollary 3.1 is a necessary condition for the existence of a common  $\mathbb{Q}$ -form of  $\{\mathbf{G}_\ell^\circ \subset \text{GL}_{V_\ell}\}_\ell$ .*

#### 4. SEMISIMPLICITY

In this section, we give two examples of  $\{\Phi_\ell\}$  such that the hypothesis of Theorem 3 holds. It suffices to show by the lemma below that the restriction of  $\phi_\ell$  to a normal subgroup of  $\pi_1^{et}(X_{\bar{\mathbb{F}}_q})$  (i.e., by taking a connected Galois étale cover of  $X$ ) is semisimple for  $\ell \gg 0$ .

**Lemma 4.1.** [HL15b, Lemma 3.6] *Let  $F$  be a field,  $G$  a finite group,  $H$  a normal subgroup of  $G$  such that  $[G : H]$  is non-zero in  $F$ , and  $V$  a finite dimensional  $F$ -representation of  $G$ . Then  $V$  is semisimple if and only if its restriction to  $H$  is so.*

**Example 1.** Suppose the fibers of  $f : Y \rightarrow X$  are curves or abelian varieties. Then the hypothesis of Theorem 3 holds.

*Proof.* **Step I.** When  $X$  is a curve,  $\phi_\ell$  is factored through by a Galois representation of  $K(X)$ , the function field of  $X$ . When the fibers of  $f$  are abelian varieties, the conclusion follows directly from the Tate conjecture of abelian varieties over function fields [Za74a, Za74b] (see also [FW84, Ch. VI§3],[LP95, Thm. 3.1(iii)]). When the fibers are curves, the conclusion follows from above and the fact that a smooth curve and its Jacobian variety have isomorphic  $H^1$  representations.

**Step II.** For general  $X$ , we may first assume  $\mathbf{G}_\ell^{\text{geo}}$  is connected for all  $\ell$  by taking a connected Galois étale cover [LP95, Prop. 2.2(i)]. By [Ka99] and [Ch04] (see Step V of Theorem 2), there exists a smooth geometrically connected curve  $C$  of  $X$  such that the algebraic geometric monodromy group associated to  $Y \times_X C \rightarrow C$  is also equal to  $\mathbf{G}_\ell^{\text{geo}}$  for all  $\ell$ . Denote the images of  $\pi_1^{\text{et}}(C_{\bar{\mathbb{F}}_q})$  and  $\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})$  in respectively  $\text{GL}(V_\ell)$  and  $\text{GL}(\bar{V}_\ell)$  by

$$\begin{aligned}\Lambda_\ell^{\text{geo}} &\subset \Gamma_\ell^{\text{geo}} \subset \text{GL}(V_\ell); \\ \bar{\Lambda}_\ell^{\text{geo}} &\subset \bar{\Gamma}_\ell^{\text{geo}} \subset \text{GL}(\bar{V}_\ell).\end{aligned}$$

We may assume  $\bar{\Gamma}_\ell^{\text{geo}}$  is generated by its order  $\ell$  elements for  $\ell \gg 0$  by [CT15]. Since  $\pi^{-1}(\Lambda_\ell^{\text{geo}}) \subset \pi^{-1}(\Gamma_\ell^{\text{geo}})$  are compact subgroups of  $\mathbf{G}_\ell^{\text{sc}}(\mathbb{Q}_\ell)$  and  $\pi^{-1}(\Lambda_\ell^{\text{geo}})$  is hyperspecial maximal compact in  $\mathbf{G}_\ell^{\text{sc}}(\mathbb{Q}_\ell)$  for  $\ell \gg 0$  by Step I and Theorem 3, we have  $\pi^{-1}(\Lambda_\ell^{\text{geo}}) = \pi^{-1}(\Gamma_\ell^{\text{geo}})$  for  $\ell \gg 0$ . Hence, the index  $[\Gamma_\ell^{\text{geo}} : \Lambda_\ell^{\text{geo}}]$  is bounded by some constant  $C$  (depending on  $n = \dim V_\ell$ ) for  $\ell \gg 0$ . It follows that  $[\bar{\Gamma}_\ell^{\text{geo}} : \bar{\Lambda}_\ell^{\text{geo}}] \leq C$  for  $\ell \gg 0$  via the reduction map  $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_\ell)) \rightarrow \text{GL}(\bar{V}_\ell)$ . This implies that the order  $\ell$  elements of  $\bar{\Gamma}_\ell^{\text{geo}}$  belong to  $\bar{\Lambda}_\ell^{\text{geo}}$  when  $\ell \gg C$ . Since  $\bar{\Gamma}_\ell^{\text{geo}}$  is generated by its order  $\ell$  elements and  $\bar{\Lambda}_\ell$  is semisimple on  $\bar{V}_\ell$  for  $\ell \gg 0$ ,  $\bar{\Gamma}_\ell^{\text{geo}}$  is semisimple on  $\bar{V}_\ell$  for  $\ell \gg 0$ .  $\square$

**Example 2.** Identify  $\Gamma_\ell^{\text{geo}}$  as a subgroup of  $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_\ell)) = \text{GL}_n(\mathbb{Z}_\ell)$  for  $\ell \gg 0$ . Suppose there exists a connected semisimple subgroup  $\mathbf{G} \subset \text{GL}_{n,\mathbb{Q}}$  such that  $(\mathbf{G}_\ell^{\text{geo}})^\circ = \mathbf{G} \times \mathbb{Q}_\ell$  in  $\text{GL}_{n,\mathbb{Q}_\ell}$  and

$$\Gamma_\ell^{\text{geo}} \cap (\mathbf{G}_\ell^{\text{geo}})^\circ \subset \mathbf{G}(\mathbb{Z}_\ell) \subset \text{GL}_n(\mathbb{Z}_\ell)$$

for  $\ell \gg 0$ . Then the hypothesis of Theorem 3 holds.

*Proof.* **Step I.** By taking a connected Galois étale cover, we may assume  $\mathbf{G}_\ell^{\text{geo}}$  is connected for all  $\ell$  [LP95, Prop. 2.2(i)] and  $\bar{\Gamma}_\ell^{\text{geo}} = (\bar{\Gamma}_\ell^{\text{geo}})^+$  for  $\ell \gg 0$  [CT15]. The closed subgroup  $\mathbf{G} \subset \text{GL}_{n,\mathbb{Q}}$  can be extended to a closed subgroup scheme  $\mathbf{G}_{\mathbb{Z}[\frac{1}{N}]} \subset \text{GL}_{n,\mathbb{Z}[\frac{1}{N}]}$  smooth over  $\mathbb{Z}[\frac{1}{N}]$  for some sufficiently divisible integer  $N$ . Let  $\mathbf{G}_{\mathbb{F}_\ell} \subset \text{GL}_{n,\mathbb{F}_\ell}$  be the base change to  $\mathbb{F}_\ell$  for  $\ell \gg 0$ . Since  $\bar{\Gamma}_\ell^{\text{geo}} \subset \mathbf{G}_{\mathbb{F}_\ell}$  for  $\ell \gg 0$ , we obtain

(32)

$$\dim_{\bar{\mathbb{Q}}_\ell}(V_\ell^{\otimes m} \otimes \bar{\mathbb{Q}}_\ell)^{\mathbf{G}} \stackrel{\text{Lem. 2.1}}{\leq} \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\mathbf{G}_{\mathbb{F}_\ell}} \leq \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\Gamma}_\ell^{\text{geo}}}$$

for  $\ell \gg 0$ . Since  $\dim_{\mathbb{Q}_\ell}(V_\ell^{\otimes m})^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})} = \dim_{\bar{\mathbb{Q}}_\ell}(V_\ell^{\otimes m} \otimes \bar{\mathbb{Q}}_\ell)^{\mathbf{G}}$  as  $\Gamma_\ell^{\text{geo}}$  is Zariski dense in  $\mathbf{G}$ , we obtain

$$(33) \quad \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\mathbf{G}_{\mathbb{F}_\ell}} = \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\Gamma}_\ell^{\text{geo}}}$$

for  $\ell \gg 0$  by (32) and Theorem 2. Since  $\mathbf{G}_{\mathbb{F}_\ell}$  is connected semisimple for  $\ell \gg 0$ , the natural representation  $i_\ell : \mathbf{G}_{\mathbb{F}_\ell} \rightarrow \text{GL}(\bar{V}_\ell \otimes \bar{\mathbb{F}}_\ell)$  is semisimple

for  $\ell \gg 0$  [La95b]. Hence, it suffices to prove that for all  $\ell \gg 0$ , the restriction of any irreducible  $\bar{\mathbb{F}}_\ell$ -subrepresentation  $W_{\bar{\mathbb{F}}_\ell}$  of  $i_\ell$  to  $\bar{\Gamma}_\ell^{\text{geo}}$  is still irreducible as  $\bar{\Gamma}_\ell^{\text{geo}} \subset \mathbf{G}_{\bar{\mathbb{F}}_\ell}$ .

**Step II.** Suppose  $W_{\bar{\mathbb{F}}_\ell}$  is a direct summand of  $i_\ell$ . Then for any  $m \in \mathbb{N}$ , we have

$$(34) \quad \dim_{\bar{\mathbb{F}}_\ell}(W_{\bar{\mathbb{F}}_\ell}^{\otimes m})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}} = \dim_{\bar{\mathbb{F}}_\ell}(W_{\bar{\mathbb{F}}_\ell}^{\otimes m})^{\bar{\Gamma}_\ell^{\text{geo}}}$$

when  $\ell$  is sufficiently large by (33). Suppose  $\bar{\Gamma}_\ell^{\text{geo}}$  is not irreducible on  $W_{\bar{\mathbb{F}}_\ell}$ . Then there exists a  $k$ -dimensional subrepresentation  $U_{\bar{\mathbb{F}}_\ell}$  of  $\bar{\Gamma}_\ell^{\text{geo}}$  and  $k < \dim W_{\bar{\mathbb{F}}_\ell} \leq n$  holds. By (34) and (the proof of) Corollary 2.3,

$$(35) \quad \dim_{\bar{\mathbb{F}}_\ell}(\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}} = \dim_{\bar{\mathbb{F}}_\ell}(\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\bar{\Gamma}_\ell^{\text{geo}}}$$

holds when  $\ell \gg 0$ . Since  $\bar{\Gamma}_\ell^{\text{geo}} \subset \mathbf{G}_{\bar{\mathbb{F}}_\ell}$  for  $\ell \gg 0$ ,

$$(36) \quad (\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}} = (\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\bar{\Gamma}_\ell^{\text{geo}}}$$

holds when  $\ell \gg 0$ . Since  $\bar{\Gamma}_\ell^{\text{geo}}$  is generated by its order  $\ell$  elements for  $\ell \gg 0$ ,  $\text{Alt}^k U_{\bar{\mathbb{F}}_\ell}$  is one-dimensional and belongs to  $(\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\bar{\Gamma}_\ell^{\text{geo}}}$  when  $\ell \gg 0$  by construction. Thus,  $\text{Alt}^k U_{\bar{\mathbb{F}}_\ell} \subset (\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}}$  by (36) which is impossible. Indeed, let  $\{v_1, \dots, v_k\}$  be a basis of  $U_{\bar{\mathbb{F}}_\ell}$  and  $Z_{\bar{\mathbb{F}}_\ell} \neq 0$  a complement of  $U_{\bar{\mathbb{F}}_\ell}$  in  $W_{\bar{\mathbb{F}}_\ell}$ . Since  $\mathbf{G}_{\bar{\mathbb{F}}_\ell}$  is irreducible on  $W_{\bar{\mathbb{F}}_\ell}$ , there exists  $x \in \mathbf{G}_{\bar{\mathbb{F}}_\ell}(\bar{\mathbb{F}}_\ell)$  that does not preserve  $U_{\bar{\mathbb{F}}_\ell}$ . Then we have the following equations

$$x \cdot v_1 = u_1 + z_1$$

$$x \cdot v_2 = u_2 + z_2$$

$$\vdots$$

$$x \cdot v_k = u_k + z_k,$$

where the notation is defined so that  $u_i \in U_{\bar{\mathbb{F}}_\ell}$  and  $z_i \in Z_{\bar{\mathbb{F}}_\ell}$  for  $1 \leq i \leq k$ . We may assume  $\{z_1, \dots, z_h\}$  is a non-empty maximal linearly independent subset of  $\{z_1, \dots, z_k\}$ . If  $\text{Alt}^k U_{\bar{\mathbb{F}}_\ell} \subset (\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}}$ , then

$$(37) \quad x \cdot (v_1 \wedge \cdots \wedge v_k) = (u_1 + z_1) \wedge \cdots \wedge (u_k + z_k) \in \text{Alt}^k U_{\bar{\mathbb{F}}_\ell}.$$

Since  $z_1 \neq 0$ , we obtain  $k > 1$  by (37). Since we have the decomposition

$$(38) \quad \text{Alt}^k(U_{\bar{\mathbb{F}}_\ell} \oplus Z_{\bar{\mathbb{F}}_\ell}) = \bigoplus_{i+j=k} \text{Alt}^i U_{\bar{\mathbb{F}}_\ell} \otimes \text{Alt}^j Z_{\bar{\mathbb{F}}_\ell},$$

we have  $z_1 \wedge \cdots \wedge z_k = 0$  by (37), which is the same as  $h < k$ . We may assume  $z_{h+1}, \dots, z_k$  are all equal to zero by the fact that  $\{z_1, \dots, z_h\}$  is a maximal linearly independent subset of  $\{z_1, \dots, z_k\}$  and replacing  $\{v_1, \dots, v_h, v_{h+1}, \dots, v_k\}$  with a suitable basis  $\{v_1, \dots, v_h, v'_{h+1}, \dots, v'_k\}$ . It follows that  $\{z_1, \dots, z_h, u_{h+1}, \dots, u_k\}$  is linearly independent because  $x$  is

invertible. Therefore,  $u_{h+1} \wedge \cdots \wedge u_k \wedge z_1 \wedge \cdots \wedge z_h$  is non-zero, which is absurd by  $z_{h+1} = \cdots = z_k = 0$ , (37), and (38). This implies that  $\bar{\Gamma}_\ell^{\text{geo}}$  cannot have a  $k$ -dimensional subrepresentation of  $W_{\bar{\mathbb{F}}_\ell}$  when  $\ell$  is sufficiently large. Since there are finitely many  $k$  less than  $\dim W_{\bar{\mathbb{F}}_\ell} \leq n$ , we conclude that  $\bar{\Gamma}_\ell^{\text{geo}}$  is irreducible on  $W_{\bar{\mathbb{F}}_\ell}$  and thus semisimple on  $\bar{V}_\ell \otimes \bar{\mathbb{F}}_\ell$  if  $\ell$  is sufficiently large.  $\square$

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